

On the relationship of continuity and boundary regularity in PMC Dirichlet problems

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Abstract

In 1976, Leon Simon showed that if a compact subset of the boundary of a domain is smooth and has negative mean curvature, then the non-parametric least area problem with Lipschitz continuous Dirichlet boundary data has a generalized solution which is continuous on the union of the domain and this compact subset of the boundary, even if the generalized solution does not take on the prescribed boundary data. Simon's result has been extended to boundary value problems for prescribed mean curvature equations by other authors. In this note, we construct Dirichlet problems in domains with corners and demonstrate that the variational solutions of these Dirichlet problems are discontinuous at the corner, showing that Simon's assumption of regularity of the boundary of the domain is essential.

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1 Introduction

Let $n \in \mathbb{N}$ with $n \geq 2$ and suppose Ω is a bounded, open set in \mathbb{R}^n with locally Lipschitz boundary $\partial\Omega$. Fix $H \in C^2(\mathbb{R}^n \times \mathbb{R})$ such that H is bounded and $H(x, t)$ is nondecreasing in t for $x \in \Omega$. Consider the prescribed mean curvature Dirichlet problem of finding a function $f \in C^2(\Omega) \cap C^0(\overline{\Omega})$ which satisfies

$$\operatorname{div}(Tf) = H(x, f) \quad \text{in } \Omega, \quad (1)$$

$$f = \phi \quad \text{on } \partial\Omega, \quad (2)$$

where $Tf = \frac{\nabla f}{\sqrt{1+|\nabla f|^2}}$ and $\phi \in C^0(\partial\Omega)$ is a prescribed function; such a function f , if it exists, is a classical solution of the Dirichlet problem. It has been long

known (e.g. Bernstein in 1912) that some type of boundary curvature condition (which depends on H) must be satisfied in order to guarantee that a classical solution exists for each $\phi \in C^0(\partial\Omega)$ (e.g. [11, 23]). When $H \equiv 0$ and $\partial\Omega$ is smooth, this curvature condition is that $\partial\Omega$ must have nonnegative mean curvature (with respect to the interior normal direction of Ω) at each point ([11]). However, Leon Simon ([24]) has shown that if $\Gamma_0 \subset \partial\Omega$ is smooth (i.e. C^4), the mean curvature Λ of $\partial\Omega$ is negative on Γ_0 and Γ is a compact subset of Γ_0 , then the minimal hypersurface $z = f(x)$, $x \in \Omega$, extends to $\Omega \cup \Gamma$ as a continuous function, even though f may not equal ϕ on Γ . Since [24] appeared, the requirement that $H \equiv 0$ has been eliminated and the conclusion remains similar to that which Simon reached (see, for example, [1, 19, 20]).

How important is the role of boundary smoothness in the conclusions reached in [24]? We shall show, by constructing suitable domains Ω and Dirichlet data ϕ , that the existence of a “nonconvex corner” P in Γ can cause the unique generalized (e.g. variational) solution to be discontinuous at P even if $\Gamma \setminus \{P\}$ is smooth and the generalized mean curvature Λ^* (i.e. [23]) of Γ at P is $-\infty$; this shows that some degree of smoothness of Γ is required to obtain the conclusions in [24]. We shall prove the following

Theorem 1. *Let $n \in \mathbb{N}$, $n \geq 2$, and assume there exists $\lambda > 0$ such that $|H(x, t)| \leq \lambda$ for $x \in \mathbb{R}^n$ and $t \in \mathbb{R}$. Then there exist a domain $\Omega \subset \mathbb{R}^n$ and a point $P \in \partial\Omega$ such that*

- (i) $\partial\Omega \setminus \{P\}$ is smooth (C^∞),
- (ii) there is a neighborhood \mathcal{N} of P such that $\Lambda(x) < 0$ for $x \in \mathcal{N} \cap \partial\Omega \setminus \{P\}$, where Λ is the mean curvature of $\partial\Omega$, and
- (iii) $\Lambda^*(P) = -\infty$, where Λ^* is the generalized mean curvature of $\partial\Omega$,

and there exists Dirichlet boundary data $\phi \in C^\infty(\mathbb{R}^n)$ such that the minimizer $f \in BV(\Omega)$ of

$$J(u) = \int_{\Omega} |Du| + \int_{\Omega} \int_0^u H(x, t) dt \, dx + \int_{\partial\Omega} |u - \phi| d\mathcal{H}^{n-1}, \quad u \in BV(\Omega), \quad (3)$$

exists and satisfies (1), $f \in C^2(\Omega) \cap C^0(\overline{\Omega} \setminus \{P\}) \cap L^\infty(\Omega)$, $f \notin C^0(\overline{\Omega})$ and $f \neq \phi$ in a neighborhood of P in $\partial\Omega$.

Since there are certainly many examples of Dirichlet problems which have continuous solutions even though their domains fail to satisfy appropriate smoothness or boundary curvature conditions (e.g. by restricting to a smaller domain a classical solution of a Dirichlet problem on a larger domain), the question of necessary or sufficient conditions for the continuity at P of a generalized solution of a particular Dirichlet problem is of interest and the examples here suggest (to us) that a “Concus-Finn” type condition might yield necessary conditions for the continuity at P of solutions; see §5.

We view this note as being analogous to other articles (e.g. [5, 9, 10, 12]) which enhance our knowledge of the behavior of solutions of boundary value problems for prescribed mean curvature equations by constructing and analyzing specific examples. One might also compare Theorem 1 with the behavior of generalized solutions of (1)-(2) when $\partial\Omega \setminus \{P\}$ is smooth and $|H(x, \phi(x))| \leq (n-1)\Lambda(x)$ for $x \in \partial\Omega \setminus \{P\}$ (e.g. [3, 14, 15]) and with capillary surfaces (e.g. [18]).

2 Nonparametric Minimal Surfaces in \mathbb{R}^3

In this section, we will assume $n = 2$ and $H \equiv 0$; this allows us to use explicit comparison functions and illustrate our general procedure. Let Ω be a bounded, open set in \mathbb{R}^2 with locally Lipschitz boundary $\partial\Omega$ such that a point P lies on $\partial\Omega$ and there exist distinct rays l^\pm starting at P such that $\partial\Omega$ is tangent to $l^+ \cup l^-$ at P . By rotating and translating the domain, we may assume $P = (0, 1)$ and there exists a $\sigma \in (-\frac{\pi}{2}, \frac{\pi}{2})$ such that $l^- = \{(r \cos(\sigma), 1 + r \sin(\sigma)) : r \geq 0\}$, $l^+ = \{(r \cos(\pi - \sigma), 1 + r \sin(\pi - \sigma)) : r \geq 0\}$ and

$$\Omega \cap B(P, \delta) = \{(r \cos(\theta), 1 + r \sin(\theta)) : 0 < r < \delta, \theta^-(r) < \theta < \theta^+(r)\} \quad (4)$$

for some $\delta > 0$ and functions $\theta^\pm \in C^0([0, \delta))$ which satisfy $\theta^- < \theta^+$, $\theta^-(0) = \sigma$ and $\theta^+(0) = \pi - \sigma$; here $B(P, \delta)$ is the open ball in \mathbb{R}^2 centered at P of radius δ . If we set $\alpha = \frac{\pi}{2} - \sigma$, then $\alpha \in (0, \pi)$ and the angle at P in Ω of $\partial\Omega$ has size 2α . As $\sigma < 0$ goes to zero, $2\alpha > \pi$ goes to π and the (upper) region between l^- and l^+ becomes “less nonconvex” and approaches a half-plane through P . We will show that for each choice of $\sigma \in (-\frac{\pi}{2}, 0)$, there is a domain Ω as above and a choice of Dirichlet data $\phi \in C^\infty(\partial\Omega)$ such that the solution of (1)-(2) for Ω and ϕ is discontinuous at P .

Fix $\sigma \in (-\frac{\pi}{2}, -\frac{\pi}{4})$. Let ϵ be a small, fixed parameter, say $\epsilon \in (0, 0.5)$, and let $a = a(\sigma) \in (1, 2)$ be a parameter to be determined. Set $\tau = (1 + \epsilon) \cot(-\sigma)$ and $r_1 = \sqrt{\tau^2 + (1 + \epsilon)^2}$. Define $h_{2/\pi} \in C^2((0, 2) \times (-1, 1))$ by

$$h_{2/\pi}(x_1, x_2) = \frac{2}{\pi} \ln \left(\frac{\cos(\frac{\pi x_2}{2})}{\sin(\frac{\pi x_1}{2})} \right).$$

Notice that the graph of $h_{2/\pi}$ is part of Scherk’s first surface, so $\operatorname{div}(Th_{2/\pi}) = 0$ on $(0, 2) \times (-1, 1)$, and $h_{2/\pi}(t, t - 1) = 0$ for each $t \in (0, 2)$. A computation using L’Hospital’s Rule shows

$$\lim_{t \rightarrow 0^+} h_{2/\pi}((t \cos(\theta), 1 + t \sin(\theta))) = \frac{2}{\pi} \ln(-\tan(\theta)), \quad \theta \in \left(-\frac{\pi}{2}, 0\right) \quad (5)$$

Let $D = B(\mathcal{O}, 1) \cap B((\tau, -\epsilon), r_1) \cap B((-\tau, -\epsilon), r_1)$ be the intersection of three open disks and let $E \subset D$ be a strictly convex domain such that $\{x \in \partial E : x_2 < 1\}$ is a C^∞ curve, $E \cap \{x_2 \geq 0\} = D \cap \{x_2 \geq 0\}$, E is symmetric with respect to the x_2 -axis and $(0, -1) \in \partial E$; here \mathcal{O} denotes $(0, 0)$. Define

$$\Omega = B(\mathcal{O}, a) \setminus \overline{E}$$

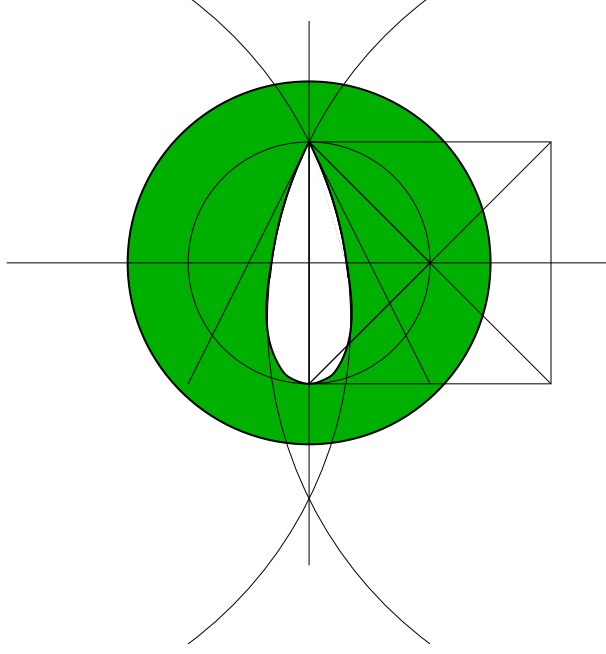


Figure 1: Ω

(see Figure 1); notice that $P \in \partial\Omega$ and (4) holds with the choice of σ above. If we set $C = \{(x_1, x_2) \in \mathbb{R}^2 : 0 < x_1 < 1, x_1 - 1 < x_2 < 1 - x_1\}$, then (5) implies $\sup_{x \in C \cap \partial E} h_{2/\pi}(x) < \infty$.

Let $m > \max\{r_1 \cosh^{-1}\left(\frac{2+\sqrt{\tau^2+\epsilon^2}}{r_1}\right), \sup_{x \in C \cap \partial E} h_{2/\pi}(x)\}$. Notice that m is independent of the parameter a . Define $\phi \in C^\infty(\partial\Omega)$ by $\phi = 0$ on $\partial B(\mathcal{O}, a)$ and $\phi = m$ on ∂E . Let f be the variational solution of (1)-(2) with ϕ as given here (e.g. [6, 8]). Since $\phi \geq 0$ on $\partial\Omega$ and $\phi > 0$ on ∂E , $f \geq 0$ in Ω (e.g. Lemma 2 (with $h \equiv 0$)) and so $f > 0$ in Ω (e.g. the Hopf boundary point lemma). Notice that $h_{2/\pi} = 0 < f$ on $\Omega \cap \partial C$ and $h_{2/\pi} < \phi$ on $C \cap \partial E = C \cap \partial\Omega$ and therefore $h_{2/\pi} < f$ on $\Omega \cap C$ (see Figure 2). Together with (5), this implies

$$\liminf_{\Omega \cap C \ni x \rightarrow P} f(x) \geq \frac{2}{\pi} \ln(\tan(-\sigma)) > 0. \quad (6)$$

Set $W = B(\mathcal{O}, a) \setminus \overline{B(\mathcal{O}, 1)}$ (see Figure 3); then $W \subset \Omega$. Define $g \in C^\infty(W) \cap C^0(\overline{W})$ by $g(x) = \cosh^{-1}(a) - \cosh^{-1}(|x|)$ and notice that the graph of g is part of a catenoid, where $g = 0$ on $\partial B(\mathcal{O}, a)$ and $g = \cosh^{-1}(a)$ on $\partial B(\mathcal{O}, 1)$. It follows from the General Comparison Principle (e.g. [4], Theorem 5.1) that $f \leq g$ on W and therefore

$$f \leq \cosh^{-1}(a) \quad \text{on } W. \quad (7)$$

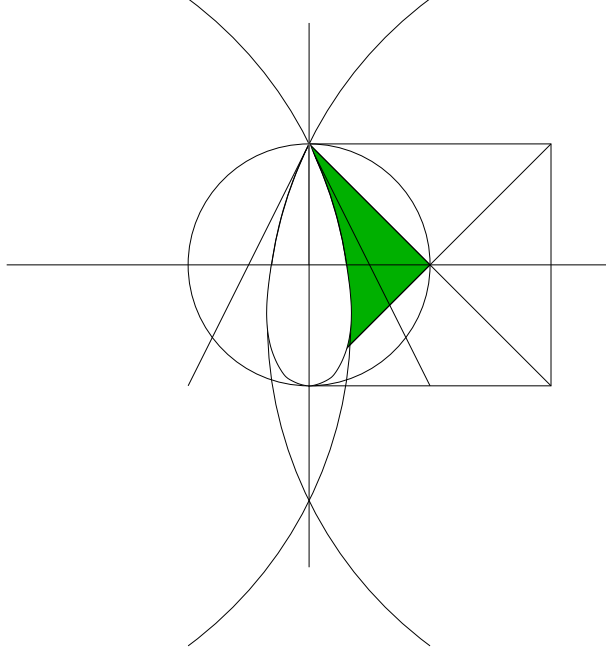


Figure 2: $\Omega \cap C$, the domain of the comparison function for (6)

If we select $a > 1$ so that $\cosh^{-1}(a) < \frac{2}{\pi} \ln(\tan(-\sigma))$, then (6) and (7) imply that f cannot be continuous at P . Notice that [24] implies $f \in C^0(\overline{\Omega} \setminus \{P\})$.

This example illustrates the procedure we shall use in §4; a somewhat similar approach was used in [5, 12, 18, 23]. The case when $\sigma \in [-\frac{\pi}{4}, 0)$ has a similar proof with the changes that D is the intersection of the open disk $B(\mathcal{O}, 1)$ with the interiors of two ellipses and a Scherk surface with rhomboidal domain ([22], pp. 70-71) is used as a comparison surface to obtain the analog of (6); the details can be found in [21].

3 Lemmata

Lemma 1. *Let Ω be a bounded open set in \mathbb{R}^n , $n \geq 2$, with locally Lipschitz boundary and let Γ be an open, C^2 subset of $\partial\Omega$. Let $\phi \in L^\infty(\partial\Omega) \cap C^{1,\beta}(\Gamma)$. Suppose $g \in C^2(\Omega) \cap L^\infty(\Omega)$ is the variational solution of (1)-(2) and $g < \phi$ on Γ . Then $\nu \equiv \frac{(\nabla g, -1)}{\sqrt{1+|\nabla g|^2}} \in C^0(\Omega \cup \Gamma)$ and $\nu \cdot \eta = 1$ on Γ , where $\eta(x) \in S^{n-1}$ is the exterior unit normal to Γ at x .*

Proof: Since g minimizes the functional J in (3) over $BV(\Omega)$, g also minimizes

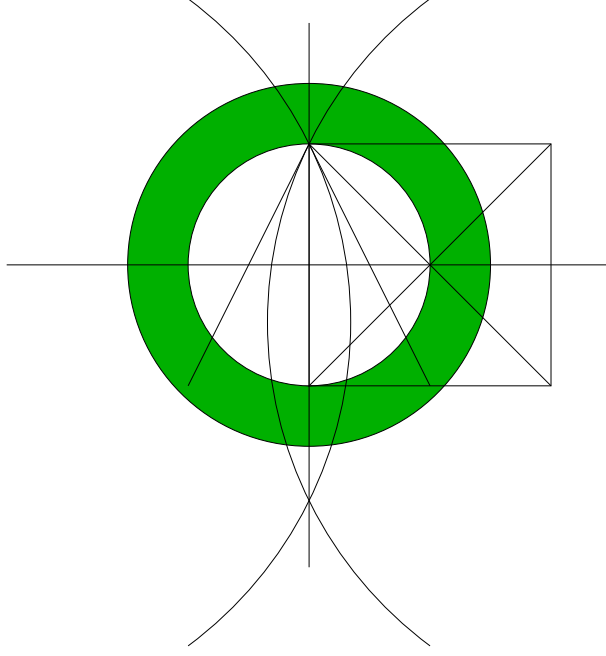


Figure 3: W , the domain of the comparison function for (7)

the functional $K(u) = J(u) - \int_{\Gamma} \phi \, d\mathcal{H}^{n-1}$. Notice

$$K(u) = \int_{\Omega} |Du| + \int_{\Omega} \int_0^u H(x, t) dt \, dx + \int_{\partial\Omega \setminus \Gamma} |u - \phi| d\mathcal{H}^{n-1} - \int_{\Gamma} u \, d\mathcal{H}^{n-1}$$

for each $u \in BV(\Omega)$ with $tr(u) \leq \phi$ on Γ ; in particular, this holds when $u = g$. Therefore, for each $x \in \Gamma$, there exists $\rho > 0$ such that $\partial\Omega \cap B_n(x, \rho) \subset \Gamma$ and the Lemma follows as in [13]. \square

Lemma 2. *Let Ω be a bounded open set in \mathbb{R}^n , $n \geq 2$, with locally Lipschitz boundary, $\phi, \psi \in L^\infty(\partial\Omega)$ with $\psi \leq \phi$ on $\partial\Omega$, $H_0 \in C^2(\Omega \times \mathbb{R})$ with $H_0(x, t)$ nondecreasing in t for $x \in \Omega$, and $H_0 \geq H$ on $\Omega \times \mathbb{R}$. Consider the boundary value problem*

$$\operatorname{div}(Tf) = H_0(x, f) \quad \text{in } \Omega, \quad (8)$$

$$f = \psi \quad \text{on } \partial\Omega. \quad (9)$$

Suppose $g \in C^2(\Omega) \cap L^\infty(\Omega)$ is the variational solution of (1)-(2) and either (i) $h \in C^2(\Omega) \cap L^\infty(\Omega)$ is the variational solution of (8)-(9) or (ii) $\psi \in C^0(\partial\Omega)$, $h \in C^2(\Omega) \cap C^0(\overline{\Omega})$ and h satisfies (8)-(9). Then $h \leq g$ in Ω .

Proof: Let $A = \{x \in \Omega : h(x) > g(x)\}$. In case (i), let $f = hI_{\Omega \setminus A} + gI_A$, where I_B is the characteristic function of a set B ; then a simple calculation

using $J(g) \leq J(f)$ shows that $J_1(f) \leq J_1(h)$ and therefore $f = h$ and $A = \emptyset$, where $J_1(u) = \int_{\Omega} |Du| + \int_{\Omega} \int_0^u H_0(x, t) dt \, dx + \int_{\partial\Omega} |u - \psi| d\mathcal{H}^{n-1}$, $u \in BV(\Omega)$, is the functional h minimizes. Case (ii) follows from Lemma 1 of [25]. \square

Lemma 3. *Let $\Omega \subset \{x \in \mathbb{R}^2 : x_2 > 0\}$ be a bounded open set, $n \in \mathbb{N}$ with $n \geq 2$ and $g \in C^2(\Omega)$. Set $\tilde{\Omega} = \{(x_1, x_2\omega) \in \mathbb{R}^n : (x_1, x_2) \in \Omega, \omega \in S^{n-2}\}$ and define $\tilde{g} \in C^2(\tilde{\Omega})$ by $\tilde{g}(x_1, x_2\omega) = g(x_1, x_2)$ for $(x_1, x_2) \in \Omega, \omega \in S^{n-2}$. Then, for $x = (x_1, \dots, x_n) = (x_1, r\omega) \in \tilde{\Omega}$ with $r = \sqrt{x_2^2 + \dots + x_n^2}$, $\omega = \frac{1}{r}(x_2, \dots, x_n)$ and $(x_1, r) \in \Omega$, we have*

$$\operatorname{div} \left(\frac{\nabla \tilde{g}}{\sqrt{1 + |\nabla \tilde{g}|^2}} \right) (x) = \operatorname{div} \left(\frac{\nabla g}{\sqrt{1 + |\nabla g|^2}} \right) (x_1, r) + \frac{n-2}{r} \frac{g_{x_2}(x_1, r)}{\sqrt{1 + |\nabla g(x_1, r)|^2}}.$$

In particular, if $H \geq 0$, $R > 0$, $\Omega \subset \{x \in \mathbb{R}^2 : x_2 \geq R\}$ and

$$\operatorname{div} \left(\frac{\nabla g}{\sqrt{1 + |\nabla g|^2}} \right) \geq H + \frac{n-2}{R} \quad \text{on } \Omega,$$

then $\operatorname{div} \left(\frac{\nabla \tilde{g}}{\sqrt{1 + |\nabla \tilde{g}|^2}} \right) \geq H$ on $\tilde{\Omega}$

Proof: Notice that $1 + |\nabla \tilde{g}|^2 = 1 + |\nabla g|^2$,

$$(1 + |\nabla \tilde{g}|^2) \triangle \tilde{g} = (1 + |\nabla g|^2) \left(\triangle g + \frac{n-2}{r} g_{x_2} \right),$$

$$\sum_{i,j=1}^n \frac{\partial \tilde{g}}{\partial x_i} \frac{\partial \tilde{g}}{\partial x_j} \frac{\partial^2 \tilde{g}}{\partial x_i \partial x_j} = \left(\frac{\partial g}{\partial x_1} \right)^2 \frac{\partial^2 g}{\partial x_1^2} + 2 \frac{\partial g}{\partial x_1} \frac{\partial g}{\partial x_2} \frac{\partial^2 g}{\partial x_1 \partial x_2} + \left(\frac{\partial g}{\partial x_2} \right)^2 \frac{\partial^2 g}{\partial x_2^2}$$

and so

$$\begin{aligned} & (1 + |\nabla \tilde{g}|^2) \triangle \tilde{g} - \sum_{i,j=1}^n \frac{\partial \tilde{g}}{\partial x_i} \frac{\partial \tilde{g}}{\partial x_j} \frac{\partial^2 \tilde{g}}{\partial x_i \partial x_j} \\ &= (1 + g_{x_2}^2) g_{x_1 x_1} - 2 g_{x_1} g_{x_2} g_{x_1 x_2} + (1 + g_{x_1}^2) g_{x_2 x_2} + \frac{n-2}{r} (1 + g_{x_1}^2 + g_{x_2}^2) g_{x_2}. \end{aligned}$$

The lemma follows from this. \square

4 The n -dimensional case

Let $B_k(x, r)$ denote the open ball in \mathbb{R}^k centered at $x \in \mathbb{R}^k$ with radius $r > 0$ and $\mathcal{O}_k = (0, \dots, 0) \in \mathbb{R}^k$, for $k \in \mathbb{N}$. Now consider $n \geq 2$ and set

$$\lambda = \sup_{(x,t) \in \mathbb{R}^n \times \mathbb{R}} |H(x, t)|;$$

if $\lambda = 0$, replace it with a positive constant. For each $a \in (0, \frac{n}{\lambda})$ and $Q \in \mathbb{R}^n$, we have

$$\int_{B_n(Q,a)} \lambda^n dx < n^n \omega_n. \quad (10)$$

By translating our problem in \mathbb{R}^n , we may (and will) assume $Q = \mathcal{O}_n$. By Proposition 1.1 and Theorem 2.1 of [7], we see that if Ω is a bounded, connected, open set in \mathbb{R}^n with Lipschitz-continuous boundary, $\overline{\Omega} \subset B_n(\mathcal{O}_n, \frac{n}{\lambda})$ and $\phi \in L^1(\partial\Omega)$, then the functional J in (3) has a minimizer $f \in BV(\Omega)$, $f \in C^2(\Omega)$ and f satisfies (1).

The proof in §4.1 consists of setting some parameters (e.g. $p, r_1, r_2, m_0, b, c, \tau, \sigma, a$), determining the domain Ω , finding different comparison functions (e.g. $g_1, g^{[u]}, k_{\pm}, k_2, k_3, k_4$), and mimicking (6) and (7) to show that the variational solution f of (1)-(2) is discontinuous at a nonconvex corner. In particular, we use a torus (i.e. j_a) to obtain (21), unduloids (i.e. k_{\pm}, k_2) to obtain (24) (an analog of (7)) and nodoids (i.e. $g_1, g^{[u]}$), unduloids (i.e. k_{\pm}, k_4) and a helicoidal function (i.e. h_2) to obtain (30) (an analog of (6)) and prove that f is discontinuous at $P = (0, p, 0, \dots, 0) \in \mathbb{R}^n \in \partial\Omega$.

4.1 Codimension 1 singular set

In this section, we will obtain a domain Ω as above and $\phi \in C^\infty(\mathbb{R}^n)$ such that $P \in \partial\Omega$, the minimizer f of (3) is discontinuous at P , $\partial\Omega \setminus T$ is smooth (C^∞) and $f \in C^2(\Omega) \cap C^0(\overline{\Omega} \setminus T)$, where T is a smooth set of dimension $n - 2$ (i.e. T has codimension 1 in $\partial\Omega$). We will use portions of nodoids, unduloids and helicoidal surfaces with constant mean curvature as comparison functions. For the convenience of the reader, we will denote functions whose graphs are subsets of nodoids with the letter g (e.g. $g_1(x_1, x_2)$), subsets of CMC helicoids with the letter h and subsets of unduloids (or onduloids) with the letter k .

Let $\mathcal{N}_1 \subset \mathbb{R}^3$ be a nodoid which is symmetric with respect the x_3 -axis and has mean curvature 1 (when \mathcal{N}_1 is oriented “inward”, so that the unit normal $\vec{N}_{\mathcal{N}_1}$ to \mathcal{N}_1 points toward the x_3 -axis at the points of \mathcal{N}_1 which are furthest from the x_3 -axis). Let $s_1 = \inf_{(x,t) \in \mathcal{N}_1} |x|$ be the inner neck size of \mathcal{N}_1 and let s_3 satisfy the condition that the unit normal to \mathcal{N}_1 is vertical (i.e. parallel to the x_3 -axis) at each point $(x, t) \in \mathbb{R}^2 \times \mathbb{R}$ of \mathcal{N}_1 at which $|x| = s_3$; then $s_1 < s_3$. Let $s_2 \in (s_1, s_3)$. (Notice that we can assume s_2/s_1 is close to s_3/s_1 if we wish.)

Let us fix $0 < p < \frac{1}{\lambda}$ and set $w = (0, p) \in \mathbb{R}^2$, $P = (0, p, 0, \dots, 0) \in \mathbb{R}^n$. Let $m_0 = \lambda/2 + (n-2)/(p/3)$. We shall assume $r_2 = s_2/m_0 < p/3$; if necessary, we increase m_0 to accomplish this. Let $r_1 = s_1/m_0$ and $r_3 = s_3/m_0$. Let $\mathcal{N} = \{(m_0)^{-1}X \in \mathbb{R}^3 : X \in \mathcal{N}_1\}$; then \mathcal{N} is a nodoid with mean curvature m_0 . Set $\Delta_1 = \{x \in \mathbb{R}^2 : r_1 < |x| < r_2\}$. Fix $b \in (0, \frac{1}{4m_0} (1 + 2m_0p - \sqrt{1 + 4m_0^2p^2}))$.

Define $g_1 \in C^\infty(\Delta_1) \cap C^0(\overline{\Delta_1})$ to be a function whose graph is a subset of \mathcal{N} on which $\vec{N}_{\mathcal{N}} = (n_1, n_2, n_3)$ satisfies $n_3 \geq 0$; then

$$\operatorname{div} \left(\frac{\nabla g_1}{\sqrt{1 + |\nabla g_1|^2}} \right) = m_0 \geq \lambda + \frac{2(n-2)}{p/3}. \quad (11)$$

By moving \mathcal{N} vertically, we may assume $g_1(x) = 0$ when $|x| = r_2$; then $g_1 > 0$ in Δ_1 . Notice that $\frac{\partial g_1}{\partial x_1}(r_1, 0) = -\infty$ and $\frac{\partial g_1}{\partial x_1}(r_2, 0) < 0$; then there exists a $\beta_0 > 0$ such that, for each $\theta \in \mathbb{R}$,

$$\frac{\partial}{\partial r}(g_1(r\Theta)) < -\beta_0 \quad \text{for } r_1 < r < r_2, \quad (12)$$

where $\Theta = (\cos(\theta), \sin(\theta))$. Fix $\beta \in (0, \beta_0)$. Let

$$0 < \tau < \min \left\{ \frac{pr_1}{\sqrt{r_2^2 - r_1^2}}, \frac{2(1-p\lambda)}{\lambda(2-p\lambda)}, \frac{b(4p-b)}{4(2p-b)} \right\}. \quad (13)$$

Consider $\sigma \in (-\frac{\pi}{2}, 0)$. Notice that the distance between L and the point $(0, p - r_2)$ is $r_2 \cos(\sigma)$, where L is the closed sector given by

$$L = \{(r \cos(\theta), p + r \sin(\theta)) : r \geq 0, \sigma \leq \theta \leq \pi - \sigma\}.$$

Define $r_4 = \sqrt{p^2 + \tau^2}$ and

$$M = B_2((\tau, 0), r_4) \cap B_2((-\tau, 0), r_4).$$

Notice that $\tau < \frac{b(4p-b)}{4(2p-b)}$ and therefore $B_2(\mathcal{O}_2, \frac{a+p}{2} - b) \subset M$ if $p < a < p + b$.

Set $\sigma = -\arctan(\tau/p)$; then $\cos(\sigma) > \frac{r_1}{r_2}$, since $\tau < \frac{p\sqrt{r_2^2 - r_1^2}}{r_1}$, and $L \cap \overline{B_2} = \emptyset$, where $B_2 = B_2((0, p - r_2), r_1)$. Therefore there exists a $\delta_1 > 0$ such that if $u = (u_1, u_2) \in \partial B_2(\mathcal{O}_2, p)$ with $|u - w| < \delta_1$, then

$$B_2\left(\frac{p - r_2}{p}u, r_1\right) \subset M. \quad (14)$$

Since $\tau < \frac{2(1-p\lambda)}{\lambda(2-p\lambda)}$, we have $\tau - (\frac{2}{\lambda} - r_4) < -p$ and so $B_2(\mathcal{O}_2, p) \subset B_2((\tau, 0), \frac{2}{\lambda} - r_4)$ (see Figure 8 (b)). Notice that

$$M \setminus \{(0, \pm p)\} = \{(r \cos(\theta), p + r \sin(\theta)) : 0 < r < 2p, \theta^-(r) < \theta < \theta^+(r)\} \quad (15)$$

for some functions $\theta^\pm \in C^0([0, \delta))$ which satisfy $\theta^- < \theta^+$, $\theta^-(0) = -\pi - \sigma$ and $\theta^+(0) = \sigma$.

Let $a > p$ and set

$$\mathcal{T} = \left\{ \left(\left(\frac{a+p}{2} + b \cos v \right) \cos u, \left(\frac{a+p}{2} + b \cos v \right) \sin u, b \sin v + c \right) : (u, v) \in R \right\},$$

where $R = [0, 2\pi] \times [-\pi, 0]$ and $0 < c < b$; since $b < \frac{1}{4m_0} \left(1 + 2m_0p - \sqrt{1 + 4m_0^2p^2} \right)$, we see that $\frac{(a+p)/2 - 2b}{4b((a+p)/2 - b)} > m_0$ for all $a \geq p$. We shall assume

$$a \in (p, \min\{p + b, 1/\lambda\}) \quad (16)$$

and $c = \sqrt{b^2 - \left(\frac{a-p}{2}\right)^2}$. Notice that \mathcal{T} is the lower half of a torus whose mean curvature (i.e. one half of the trace of the shape operator) at each point is

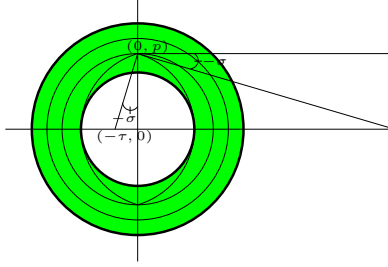


Figure 4: The domain of j_a

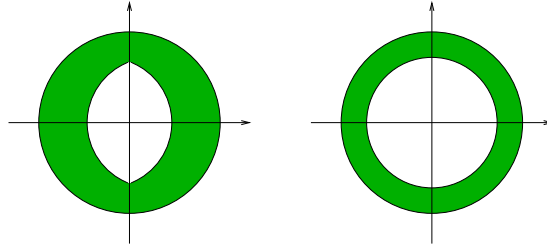


Figure 5: (a) $\Pi_{1,j}(\Omega)$ for $2 \leq j \leq n$ (b) $\Pi_{i,j}(\Omega)$ for $2 \leq i < j \leq n$

greater than m_0 . Let \mathcal{T} be the graph of a function j_a over $\Delta_a = \{x \in \mathbb{R}^2 : \frac{a+p}{2} - b \leq |x| \leq \frac{a+p}{2} + b\}$; then $j_a(x) = 0$ on $|x| = a$ and $|x| = p$, $j_a(x) < 0$ on $p < |x| < a$ and $j_a(x) > 0$ on $\frac{a+p}{2} - b \leq |x| < p$ and $a < |x| \leq \frac{a+p}{2} + b$ for $x \in \mathbb{R}^2$. Notice that $|j_a(x)| < \frac{1}{2m_0}$ for all $x \in \Delta_a$.

Set

$$\Omega = B_n(\mathcal{O}_n, a) \setminus \overline{\mathcal{M}}, \quad (17)$$

where $\mathcal{M} = \tilde{M} = \{(x_1, x_2, \omega) \in \mathbb{R}^n : (x_1, x_2) \in M, \omega \in S^{n-2}\}$. If we define $\Pi_{i,j}(A) = \{(x_i, x_j) : (x_1, \dots, x_n) \in A, x_k = 0 \text{ for } k \neq i, j\}$ for $A \subset \mathbb{R}^n$ and $1 \leq i < j \leq n$, then $\Pi_{1,j}(\Omega) = B_2(\mathcal{O}_2, a) \setminus \overline{M}$ for $2 \leq j \leq n$ and $\Pi_{i,j}(\Omega) = B_2(\mathcal{O}_2, a) \setminus \overline{B_2(\mathcal{O}_2, 1)}$ for $2 \leq i < j \leq n$ (see Figure 5).

We wish to select a helicoidal surface in \mathbb{R}^3 (e.g. [2]) with constant mean curvature m_0 , axis $\{w\} \times \mathbb{R}$ and pitch $-\beta$ (recall $-\beta \in (-\beta_0, 0)$), which we will denote \mathcal{S} ; then, for each $t \in \mathbb{R}$, $k_t(\mathcal{S}) = \mathcal{S}$, where $k_t : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ is the helicoidal motion given by $k_t(x_1, x_2, x_3) = (l_t(x_1, x_2), x_3 - \beta t)$ with $l_t : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ given by

$$l_t(x_1, x_2) = (x_1 \cos(t) + (x_2 - p) \sin(t), p - x_1 \sin(t) + (x_2 - p) \cos(t)).$$

Set $c_0 = \frac{1}{4}\beta\sigma < 0$; by vertically translating \mathcal{S} , we may assume that there is an open c_0 -level curve \mathcal{L}_0 of \mathcal{S} with endpoints $w = (0, p)$ and $b = (b_1, b_2)$ such that $\mathcal{L}_0 \subset (0, \infty) \times \mathbb{R}$, $\mathcal{L} = \overline{\mathcal{L}_0}$ is tangent to the (horizontal) line $\mathbb{R} \times \{p\}$ at w and the slope m_v of the tangent line to \mathcal{L} at v satisfies $|m_v| < \tan(-\sigma/5)$ for each

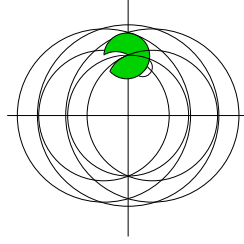


Figure 6: \mathcal{R}

$v \in \mathcal{L}_0$; then $\mathcal{L} \times \{c_0\} \subset \mathcal{S}$ and the curves $l_t(\mathcal{L}_0)$, $-\frac{7\pi}{8} < t < \frac{7\pi}{8}$, are mutually disjoint. Notice that the set

$$\mathcal{R} = \{l_t(\mathcal{L}_0) : -\frac{7\pi}{8} < t < \frac{7\pi}{8}\} = \bigcup_{-\frac{7\pi}{8} < t < \frac{7\pi}{8}} l_t(\mathcal{L}_0)$$

is an open subset of $\mathbb{R}^2 \setminus ((-\infty, 0] \times \{p\})$ (see Figure 6), $w \in \overline{\mathcal{R}}$ and \mathcal{S} implicitly defines the smooth function h_2 on \mathcal{R} given by $h_2(x) = \frac{\beta}{4}(\sigma - 4t)$ if $x \in l_t(\mathcal{L}_0)$ for some $t \in (-\pi/2, \pi/2)$. Notice that $B_2(w, b_1) \cap \{x_1 > 0\} \subset \mathcal{R}$. Now $l_t(\mathcal{L}_0) \cap M = \emptyset$ for $t \in (3\sigma/4, \sigma/4)$ and, by making $b_1 > 0$ sufficiently small, we may assume that

$$l_t(\mathcal{L}_0) \subset B_2(\mathcal{O}_2, p) \setminus M \quad \text{for each } t \in (3\sigma/4, \sigma/4). \quad (18)$$

Notice that $h_2 < \frac{\beta(2\sigma^2 - \pi)}{8\sigma}$ on $l_t(\mathcal{L}_0)$ for $-\frac{\pi}{2} < t < \frac{7\pi}{8}$.

Let us fix $u = (u_1, u_2) \in \partial B_2(\mathcal{O}_2, p)$ such that $|u - w| < \min\{\delta_1, b_1\}$ and $u_1 > 0$. Then there exists $\theta_u \in (0, \pi/2)$ such that $u = (p \cos(\theta_u), p \sin(\theta_u))$. Define $g^{[u]}(x) = g_1\left(x + \frac{r_2 - p}{p}u\right)$ and notice that $g^{[u]}(u) = g_1\left(\frac{r_2}{p}u\right) = 0$, since $|\frac{r_2}{p}u| = r_2$. Note that the domain

$$\mathcal{D}^{[u]} = \{x + \frac{p - r_2}{p}u : x \in \Delta_1\} = B_2\left(\frac{p - r_2}{p}u, r_2\right) \setminus \overline{B_2\left(\frac{p - r_2}{p}u, r_1\right)}$$

of $g^{[u]}$ is contained in $B_2(\mathcal{O}_2, p)$ since $\partial B_2\left(\frac{p - r_2}{p}u, r_2\right)$ and $\partial B_2(\mathcal{O}_2, p)$ are tangent circles at u and $r_2 < p$ (see Figure 7). Notice that

$$h_2(r \cos(\theta_u), r \sin(\theta_u)) < g^{[u]}(r \cos(\theta_u), r \sin(\theta_u)) \quad (19)$$

when $p - r_2 + r_1 \leq r \leq p$, because $h_2(u) < 0 = g^{[u]}(u)$, $\beta < \beta_0$ and (12) holds.

Let

$$\mathcal{N}_\pm \subset \{x \in \mathbb{R}^2 : r_4 \leq |(x_1 \pm \tau, x_2)| \leq \frac{2}{\lambda} - r_4\} \times \mathbb{R}$$

be unduloids in \mathbb{R}^3 with mean curvature $\lambda/2$ such that $\{(\mp\tau, 0)\} \times \mathbb{R}$ are the respective axes of symmetry; the minimum and maximum radii (or “neck” and

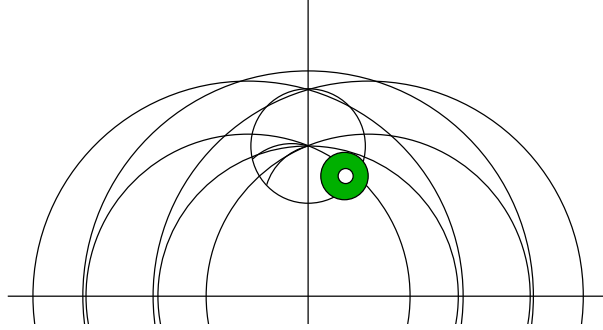


Figure 7: $\mathcal{D}^{[u]}$; $\Omega \cap \tilde{\mathcal{D}}^{[u]}$ is the domain of the comparison function for (28)

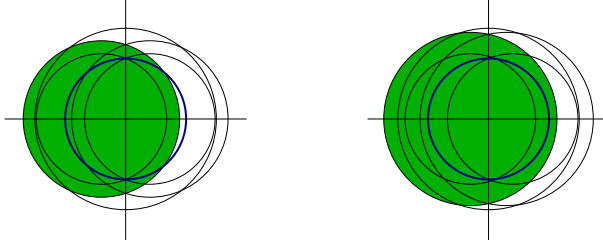


Figure 8: (a) $B_2(\mathcal{O}_2, p) \not\subset B_2((-\tau, 0), \frac{2}{\lambda} - r_4)$ (b) $B_2(\mathcal{O}_2, p) \subset B_2((-\tau, 0), \frac{2}{\lambda} - r_4)$

“waist” sizes) of both unduloids are r_4 and $\frac{2}{\lambda} - r_4$ respectively. Set

$$\Delta_{\pm} = B_2\left((\mp\tau, 0), \frac{2}{\lambda} - r_4\right) \setminus \overline{B_2((\mp\tau, 0), r_4)}$$

and define $k_{\pm} \in C^{\infty}(\Delta_{\pm})$ so that the graphs of k_{\pm} are subsets of \mathcal{N}_{\pm} respectively,

$$\operatorname{div}(Tk_{\pm}) = -\lambda \quad \text{in } \Delta_{\pm},$$

$\frac{\partial}{\partial r}(k_{\pm}((\mp p, 0) + r\Theta))|_{r=r_4} = -\infty$ and $\frac{\partial}{\partial r}(k_{\pm}((\mp p, 0) + r\Theta))|_{r=\frac{2}{\lambda}-r_4} = -\infty$ for each $\theta \in \mathbb{R}$, where $\Theta = (\cos(\theta), \sin(\theta))$. We may vertically translate \mathcal{N}_{\pm} so that $k_{\pm}(x) = 0$ for $x \in \mathbb{R}^2$ with $|(x_1 \pm \tau, x_2)| = \frac{2}{\lambda} - r_4$. Notice that $k_+(0, p) = k_-(0, p) = \sup_{\Delta_+} k_+ = \sup_{\Delta_-} k_-$.

Let $\mathcal{N} \subset \{x \in \mathbb{R}^2 : p \leq |x| \leq \frac{2}{\lambda} - p\} \times \mathbb{R}$ be an unduloid with mean curvature $\lambda/2$ such that the x_3 -axis is the axis of symmetry and the minimum and maximum radii (or “neck” and “waist” sizes) are p and $\frac{2}{\lambda} - p$ respectively. Set $\Delta_2 = B_2(\mathcal{O}_2, \frac{2}{\lambda} - p) \setminus \overline{B_2(\mathcal{O}_2, p)}$ and define $k_2 \in C^{\infty}(\Delta_2)$ so that the graph of k_2 is a subset of \mathcal{N} , $\operatorname{div}(Tk_2) = -\lambda$ in Δ_2 , $\frac{\partial}{\partial r}(k_2(r\Theta))|_{r=p} = -\infty$ and $\frac{\partial}{\partial r}(k_2(r\Theta))|_{r=\frac{2}{\lambda}-p} = -\infty$ for each $\theta \in \mathbb{R}$, where $\Theta = (\cos(\theta), \sin(\theta))$.

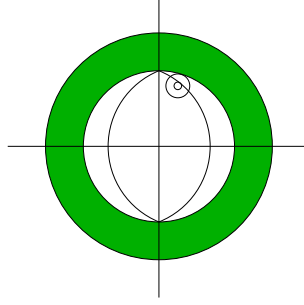


Figure 9: $B_2(\mathcal{O}_2, a) \setminus \overline{B_2(\mathcal{O}_2, p)}$: (22)

Define $\phi \in C^\infty(\mathbb{R}^n)$ so that $\phi = 0$ on $\partial B_n(\mathcal{O}_n, a)$ and $\phi = m$ on $\partial \mathcal{M}$, where

$$m > \max\{g_1(0, r_1), \frac{1}{2m_0}, k_+(0, r_4 - \tau) + k_2(0, p) - k_2\left(0, \frac{2}{\lambda} - p\right)\}; \quad (20)$$

recall then that $m > j_a\left(\frac{a+p}{2} - b\right)$. Let f be the variational solution of (1)-(2) with Ω and ϕ as given here; that is, let f minimize the functional given in (3) and notice that the existence of f follows from (10), (16), §1.D. of [7] and [6, 8]. (Notice that there exists $w : B_2(\mathcal{O}_2, a) \setminus M \rightarrow \mathbb{R}$ such that $f = \tilde{w}$.) The comparison principle implies $j_a(x) \leq f(x)$ for $x \in \Omega$ and so $f(x) \geq j_a(x) \geq 0$ if $x \in \Omega$ with $|x| \leq p$ (recall (16) holds). In particular,

$$f(x) \geq 0 \quad \text{when } x \in \Omega \text{ with } |x| \leq p. \quad (21)$$

Set $W = \left(B_2(\mathcal{O}_2, a) \setminus \overline{B_2(\mathcal{O}_2, p)}\right) \times \mathbb{R}^{n-2}$. Now

$$\Omega \subset B_2(\mathcal{O}_2, a) \times \mathbb{R}^{n-2} \subset B_2\left(\mathcal{O}_2, \frac{2}{\lambda} - p\right) \times \mathbb{R}^{n-2}$$

(see Figure 9). Define $k_3(x) = k_2(x_1, x_2) - k_2(0, a)$ for $x = (x_1, x_2, \dots, x_n) \in W$. Notice that $f = 0 \leq k_3$ on $\overline{W} \cap \partial B_n(\mathcal{O}_n, a)$,

$$\operatorname{div}(Tf) = H(x, f(x)) \geq -\lambda = \operatorname{div}(Tk_3) \quad \text{in } \Omega \cap W$$

and $\frac{\partial}{\partial r}(k_2(r\Theta))|_{r=p} = -\infty$ (so that $\lim_{W \ni y \rightarrow x} Tk_3(y) \cdot \xi(x) = 1$ for $x \in \partial B_2(\mathcal{O}_2, p) \times \mathbb{R}^{n-2}$, where ξ is the unit exterior normal to ∂W). The general comparison principle (e.g. [4], Theorem 5.1) then implies

$$f \leq k_3 \quad \text{in } \Omega \cap W \quad (22)$$

and, in particular,

$$\limsup_{\Omega \cap W \ni y \rightarrow x} f(y) \leq k_3(x) \quad \text{for } x \in \partial \Omega \cap \overline{W} \quad (23)$$

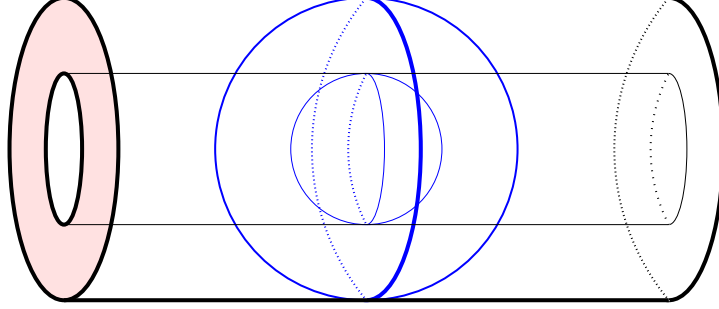


Figure 10: (23): W and $B_n(\mathcal{O}_n, a) \setminus \overline{B_n(\mathcal{O}_n, p)}$ when $n = 3$

(see Figure 10). By rotating the axis of symmetry of W through all lines in \mathbb{R}^n containing \mathcal{O}_n (or, equivalently, keeping W fixed and rotating Ω about \mathcal{O}_n), we see that

$$\sup\{f(x) : x \in B_n(\mathcal{O}_n, a) \setminus \overline{B_n(\mathcal{O}_n, p)}\} \leq k_2(0, p) - k_2(0, a). \quad (24)$$

Now define $k_4 \in C^\infty(\Delta_+ \times \mathbb{R}^{n-2}) \cap C^0(\overline{\Delta_+} \times \mathbb{R}^{n-2})$ by

$$k_4(x) = k_+(x_1, x_2) + k_2(0, p) - k_2(0, a), \quad x = (x_1, x_2, \dots, x_n) \in \overline{\Delta_+} \times \mathbb{R}^{n-2}.$$

Combining (1) and (24) with the facts that $\operatorname{div}(Tk_4) = -\lambda$ in $\Delta_+ \times \mathbb{R}^{n-2}$ and $\lim_{\Delta_+ \times \mathbb{R}^{n-2} \ni y \rightarrow x} Tk_4(y) \cdot \xi_+(x) = 1$ for $x \in \partial B_2((-\tau, 0), r_4) \times \mathbb{R}^{n-2}$, where ξ_+ is the inward unit normal to $\partial B_2((-\tau, 0), r_4) \times \mathbb{R}^{n-2}$, we see that

$$f \leq k_4 \quad \text{in } \Omega \cap (\Delta_+ \times \mathbb{R}^{n-2}). \quad (25)$$

(If Figure 8 (a) held, then (25) would not be valid.) Now let $L : \mathbb{R}^n \rightarrow \mathbb{R}^n$ be any rotation about \mathcal{O}_n which satisfies $L(\Omega) = \Omega$, notice that $f \circ L$ satisfies (1)-(2) and apply the previous argument to obtain $f \circ L \leq k_4$ in $\Omega \cap (\Delta_+ \times \mathbb{R}^{n-2})$ and therefore

$$\sup\{f(x) : x \in \partial\mathcal{M}\} \leq k_4(p, 0) < m. \quad (26)$$

From Lemma 1, we see that the downward unit normal to the graph of f , N_f , satisfies $N_f = (\nu, 0)$ on $\partial\mathcal{M} \setminus \{(0, p\omega) : \omega \in S^{n-2}\}$ and

$$\lim_{\Omega \ni y \rightarrow x} Tf(y) \cdot \nu(x) = 1 \quad \text{for } x \in \partial\mathcal{M} \setminus \{(0, p\omega) : \omega \in S^{n-2}\}. \quad (27)$$

Let us write $B = B_2\left(\frac{p-r_2}{p}u, r_2\right)$; then $\tilde{g}^{[u]} = 0 \leq f$ on $\Omega \cap \partial\tilde{B}$ and $\tilde{g}^{[u]} \leq g_1(r_1, 0) < \phi$ on $\tilde{B} \cap \partial M$. It follows from (1), (11) and Lemma 3 that

$$\tilde{g}^{[u]} < f \quad \text{on } \Omega \cap \tilde{\mathcal{D}}^{[u]} = \Omega \cap \tilde{B}. \quad (28)$$

Set $U = \{r(\cos(\theta), \sin(\theta)\omega) \in \Omega : r \in (0, p), \theta \in (0, \theta_u), \omega \in S^{n-2}\}$. If we write $\partial_1 U = \{(p\cos(\theta), p\sin(\theta)\omega) : \theta \in (0, \theta_u], \omega \in S^{n-2}\}$, $\partial_2 U = \partial\mathcal{M} \cap \partial U$

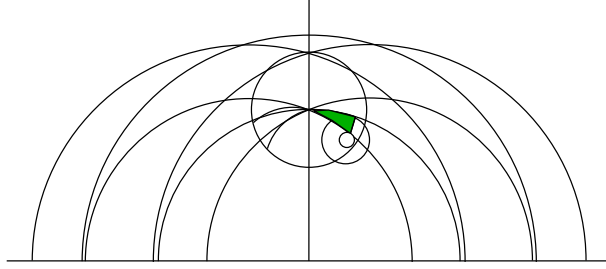


Figure 11: $A : (29)$

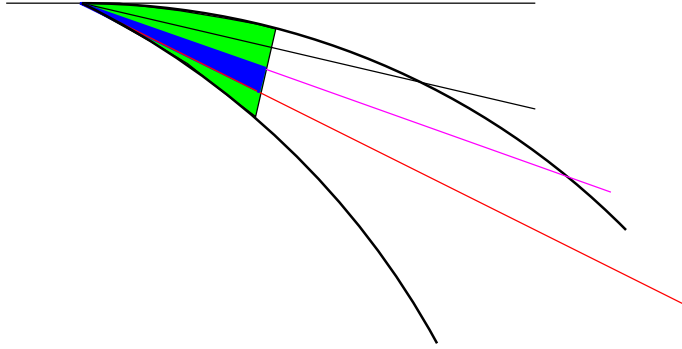


Figure 12: An illustration of \mathcal{R}_2 (blue region) and A (green and blue regions)

and $\partial_3 U = \{(r \cos(\theta_u), r \sin(\theta_u)\omega) \in \bar{\Omega} : r \in [0, p], \omega \in S^{n-2}\}$, then $\partial U = \partial_1 U \cup \partial_2 U \cup \partial_3 U$, $\tilde{h}_2 \leq 0 \leq f$ on $\partial_1 U \setminus \{P\}$ and $\tilde{h}_2 < \tilde{g}^{[u]} < f$ on $\partial_3 U$ (see (19)); then (27) and the general comparison principle imply

$$\tilde{h}_2 < f \quad \text{in} \quad U = \tilde{A}, \quad (29)$$

where $A = \{r(\cos(\theta), \sin(\theta)) \in B_2(\mathcal{O}_2, p) \setminus \bar{M} : r \in (0, p), \theta \in (0, \theta_u)\}$ (see Figure 11). Set $\mathcal{R}_2 = \bigcup_{t=3\sigma/4}^{2\sigma/4} l_t(\mathcal{L}_0)$. Now (18) implies $\tilde{\mathcal{R}}_2 \subset U$ and so

$$f > \tilde{h}_2 \geq -\frac{\beta\sigma}{4} \quad \text{on} \quad \mathcal{R}_2. \quad (30)$$

Using (24) and (30), we see that if $a \in (p, \frac{2}{\lambda} - p)$ is close enough to p , then $k_2(0, p) - k_2(0, a) < -\frac{\beta\sigma}{4}$ and therefore f cannot be continuous at P or at any point of $T = \{(0, p\omega) \in \mathbb{R}^n : \omega \in S^{n-2}\}$. Notice that $f \in C^0(\bar{\Omega} \setminus T)$ (e.g. [20]).

4.2 One singular point

In this section, we will obtain a domain Ω and $\phi \in C^\infty(\mathbb{R}^n)$ such that $P \in \partial\Omega$, the minimizer f of (3) is discontinuous at P , $\partial\Omega \setminus \{P\}$ is smooth (C^∞) and

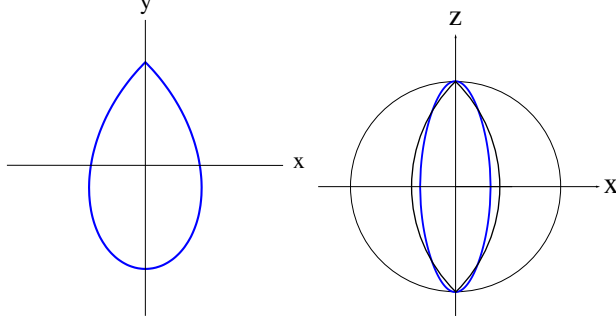


Figure 13: $X\left(\theta, \frac{\pi}{2}, 1\right)$, $X\left(\theta, \frac{1}{2} \arccos(1 - \sec(\theta) \sec(2\theta)), 1\right)$

$f \in C^0(\overline{\Omega} \setminus \{P\})$. This is accomplished by replacing \mathcal{M} by a convex set \mathcal{G} such that $\partial\mathcal{G} \setminus \{P\}$ is smooth (C^∞) and $\mathcal{G} \subset B_n(\mathcal{O}_n, p)$. We shall use the notation of §4.1 throughout this section. We assume $p \in (0, \frac{1}{\lambda})$ and set $P = (0, p, 0, \dots, 0)$. (We will no longer require Figure 8 (b) to hold.)

Let $\alpha > 1$, $n \geq 3$, and $Y : [-\frac{\pi}{2\alpha}, \frac{\pi}{2\alpha}] \times [0, \pi] \times S^{n-3} \rightarrow \mathbb{R}^n$ be defined by

$$Y(\theta, \phi, \omega) = 2 \cos(\alpha\theta) \sin(\phi) (\cos(\theta) \sin(\phi), \sin(\theta) \sin(\phi), \cos(\phi)\omega).$$

Let $F : \mathbb{R}^n \rightarrow \mathbb{R}^n$ be given by $F(x_1, \dots, x_n) = \left(\frac{x_2}{p}, \frac{1-x_1}{p}, \frac{x_3}{p}, \dots, \frac{x_n}{p}\right)$ and define $X(\theta, \phi, \omega) = F(Y(\theta, \phi, \omega))$ for $-\frac{\pi}{2\alpha} \leq \theta \leq \frac{\pi}{2\alpha}$, $0 \leq \phi \leq \pi$, $\omega \in S^{n-3}$ (see Figures 13 and Figure 14 with $n = 3$, $\alpha = 2$; the axes are labeled x, y, z for x_1, x_2, x_3 respectively). Let \mathcal{G} be the open, convex set whose boundary is the image of X ; that is,

$$\partial\mathcal{G} = \{X(\theta, \phi, \omega) : -\frac{\pi}{2\alpha} \leq \theta \leq \frac{\pi}{2\alpha}, 0 \leq \phi \leq \pi, \omega \in S^{n-3}\}.$$

Notice that $\partial\mathcal{G} \setminus \{P\}$ is a C^∞ hypersurface in \mathbb{R}^n and $\partial\mathcal{G} \subset \overline{B_n(\mathcal{O}_n, p)}$.

Let τ satisfy

$$0 < \tau < \min \left\{ \frac{pr_1}{\sqrt{r_2^2 - r_1^2}}, \frac{b(4p - b)}{4(2p - b)} \right\}.$$

Set $\sigma = -\arctan(\tau/p)$ and $\alpha = \frac{\pi}{\pi+2\sigma}$. Then the tangent cones to $\partial\mathcal{G}$ and $\partial\mathcal{M}$ at P are identical, $\cos(\sigma) > \frac{r_1}{r_2}$ and (14) holds for $u = (u_1, u_2) \in \partial B_2(\mathcal{O}_2, p)$ with $|u - w| < \delta_1$. If necessary by making $\tau > 0$ smaller, we may assume $B_n(\mathcal{O}_n, \frac{a+p}{2} - b) \subset \mathcal{G}$ if $p < a < p + b$.

Now pick $a \in (p, \min\{p + b, 1/\lambda\})$ such that $k_2(0, p) - k_2(0, a) < -\frac{\beta\sigma}{4}$, as in (30), and define

$$\Omega = B_n(\mathcal{O}_n, a) \setminus \overline{\mathcal{G}}. \quad (31)$$

Let

$$m > \max\{g_1(0, r_1), \frac{1}{2m_0}, \frac{\beta(2\sigma^2 - \pi)}{8\sigma}\}$$

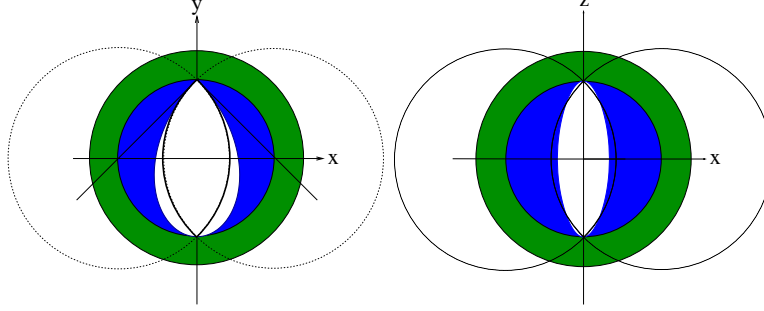


Figure 14: (a) $\Pi_{1,2}(\Omega)$ (b) $\Pi_{1,3}(\Omega)$

and define $\phi \in C^\infty(\mathbb{R}^n)$ so that $\phi = 0$ on $\partial B_n(\mathcal{O}_n, a)$ and $\phi = m$ on $\partial \mathcal{G}$ and let f be the variational solution of (1)-(2). Notice that $f \in C^2(\Omega)$ satisfies (1) and $f \in C^0(\bar{\Omega} \setminus \{P\})$ (e.g. [20]).

As in (28), let $B = B_2\left(\frac{p-r_2}{p}u, r_2\right)$. Set $U_0 = \{x \in \Omega : x \in \tilde{B}, x_1 > 0\}$ and $U = \{r(\cos(\theta), \sin(\theta)\omega) \in \Omega : r \in (0, p), \theta \in (0, \theta_u), \omega \in S^{n-2}\}$. Now $\tilde{g}^{[u]} = 0$ on $\partial U_0 \cap \partial \tilde{B}$ and $\tilde{g}^{[u]} \leq g_1(0, r_1) < m$ on $\partial U_0 \cap \partial \mathcal{G}$ and so Lemma 2, Lemma 3 and (1) imply $\tilde{g}^{[u]} \leq f$ in U_0 since f minimizes the functional in (3).

As before, set $\partial_1 U = \{(p \cos(\theta), p \sin(\theta)\omega) : \theta \in [0, \theta_u], \omega \in S^{n-2}\}$, $\partial_2 U = \partial \mathcal{G} \cap \partial U$ and $\partial_3 U = \{(r \cos(\theta_u), r \sin(\theta_u)\omega) \in \bar{\Omega} : r \in [0, p], \omega \in S^{n-2}\}$. Then $f \geq 0$ on $\partial_1 U \setminus \{P\}$, $\partial U = \partial_1 U \cup \partial_2 U \cup \partial_3 U$, $\tilde{h}_2 \leq 0 \leq f$ on $\partial_1 U$, $\tilde{h}_2 < m = \phi$ on $\partial_2 U$ and $\tilde{h}_2 < \tilde{g}^{[u]} < f$ on $\partial_3 U$; Lemma 2 implies that (30) continues to hold. Then (24) and (30) imply f is discontinuous at P since $k_2(0, p) - k_2(0, a) < -\frac{\beta\sigma}{4}$.

5 The Concus-Finn conjecture

For the moment, assume $n = 2$. In approximately 1970, Paul Concus and Robert Finn conjectured that if $\kappa \geq 0$, $\Omega \subset \mathbb{R}^2$ has a corner at $P \in \partial\Omega$ of (angular) size 2α , $\alpha \in (0, \frac{\pi}{2})$, $\gamma : \partial\Omega \setminus \{P\} \rightarrow [0, \pi]$ and $|\frac{\pi}{2} - \gamma_0| > \alpha$, where

$$\lim_{\partial\Omega \ni x \rightarrow P} \gamma(x) = \gamma_0, \quad (32)$$

then a function $f \in C^2(\Omega) \cap C^1(\bar{\Omega} \setminus \{P\})$ which satisfies

$$\operatorname{div}(Tf) = \kappa f \quad \text{in } \Omega, \quad (33)$$

$$Tf \cdot \eta = \cos(\gamma) \quad \text{on } \partial\Omega \setminus \{P\}, \quad (34)$$

must be discontinuous at P ; here $\eta(x)$ is the exterior unit normal to Ω at $x \in \partial\Omega \setminus \{P\}$. A generalization (including the replacement of (33) by (1)) of this conjecture in the case $\gamma_0 \in (0, \pi)$ was proven in [16].

In the situation above with $\alpha \in (\frac{\pi}{2}, \pi)$, the “nonconvex Concus-Finn conjecture” states that if $|\frac{\pi}{2} - \gamma_0| > \pi - \alpha$, then the capillary surface f with contact

angle γ must be discontinuous at P . A generalization (including the replacement of (33) by (1)) of this extension of the Concus-Finn conjecture in the case $\gamma_0 \in (0, \pi)$ was proven in [17]. Both [16] and [17] include the possibility of differing limiting contact angles; that is, the following limits

$$\lim_{\partial^+ \Omega \ni x \rightarrow P} \gamma(x) = \gamma_1 \quad \text{and} \quad \lim_{\partial^- \Omega \ni x \rightarrow P} \gamma(x) = \gamma_2$$

exist, $\gamma_1, \gamma_2 \in (0, \pi)$ and $\gamma_1 \neq \gamma_2$. Here $\partial^+ \Omega$ and $\partial^- \Omega$ are the two components of $\partial \Omega \setminus \{P, Q\}$, where $Q \in \partial \Omega \setminus \{P\}$. When $\gamma_1 \neq \gamma_2$, the necessary and sufficient (when $\alpha \leq \frac{\pi}{2}$) or necessary (when $\alpha > \frac{\pi}{2}$) conditions for the continuity of f at P become slightly more complicated.

The cases where $\gamma_0 = 0$, $\gamma_0 = \pi$, $\min\{\gamma_1, \gamma_2\} = 0$ and $\max\{\gamma_1, \gamma_2\} = \pi$ remain unresolved. If we suppose for a moment that the nonconvex Concus-Finn conjecture with limiting contact angles of zero or π is proven, then the discontinuity of f at P in §2 follows immediately from the fact that $f < \phi$ in a neighborhood in $\partial \Omega \setminus \{P\}$ of P since then Lemma 1 implies $\gamma_0 = 0$ and therefore $|\frac{\pi}{2} - \gamma_0| > \pi - \alpha$. In this situation (i.e. the solution f of a Dirichlet problem satisfies a zero (or π) contact angle boundary condition near P), establishing the discontinuity of f at P would be much easier and a much larger class of domains Ω with a nonconvex corner (i.e. $\alpha > \frac{\pi}{2}$) at P would have this property. For example, if Ω is a bounded locally Lipschitz domain in \mathbb{R}^2 for which (4) holds, $f \in C^2(\Omega)$ is a generalized solution of (1)-(2) (and H need not vanish) and ϕ is large enough near P (depending on H and the maximum of ϕ outside some neighborhood of P) that $f < \phi$ on $\partial \Omega \setminus \{P\}$ near P , then the fact that $\gamma_0 = 0$ (Lemma 1) together with the nonconvex Concus-Finn conjecture would imply that f is discontinuous at P .

Now consider $n \in \mathbb{N}$ with $n \geq 3$. Formulating generalizations of the Concus-Finn conjecture in the “convex corner case” (i.e. $\Omega \cap B_n(P, r) \subset \{X \in \mathbb{R}^n : (X - P) \cdot \mu > 0\}$ for some $\mu \in S^{n-1}$, $P \in \partial \Omega$ and $r > 0$) and in other cases where $\partial \Omega$ is not smooth at a point $P \in \partial \Omega$ may be complicated because the geometry of $\partial \Omega \setminus \{P\}$ is much more interesting when $n > 2$. Establishing the validity of a generalization of the Concus-Finn conjecture for solutions of (1) & (34) when $n > 2$ is probably significantly harder than doing so when $n = 2$.

Suppose we knew that a solution f of (1) & (34) is necessarily discontinuous at a “nonconvex corner” $P \in \partial \Omega$ when $\gamma_0 = 0$, where γ_0 is given by (32). In this case, a necessary condition for the continuity of f at P would be that $\limsup_{\partial \Omega \ni X \rightarrow P} T f(X) \cdot \eta(X) > 0$ and $\liminf_{\partial \Omega \ni X \rightarrow P} T f(X) \cdot \eta(X) < \pi$. Then the arguments in §4 could be made more easily and the conclusion that f is discontinuous at P would hold in a much larger class of domains Ω ; here, of course, we use the ridge point P in §4 as an example of a “nonconvex corner” of a domain in \mathbb{R}^n . The primary difficulty in proving in §4 that f is discontinuous at P is establishing (30); a more “natural” generalization of $\Omega \subset \mathbb{R}^2$ in §2 would be

$$\Omega^* = \{(x\omega_1, y, \omega_2, \dots, \omega_{n-1}) \in \mathbb{R}^n : (x, y) \in B_2(\mathcal{O}_2, a) \setminus \overline{M}, \omega \in S^{n-1}\}.$$

However, the use of Lemma 3 to help establish (30) in Ω^* is highly problematic.

On the other hand, an n -dimensional “Concus-Finn theorem” for a nonconvex conical point (e.g. $P \in \partial\Omega^*$) would only require an inequality like (26) to prove that $f < \phi$ on $\partial\Omega \setminus \{P\}$ near P and hence that f is discontinuous at P ; the replacement of (17) by (31) in order to obtain a Ω such that $\partial\Omega \setminus \{P\}$ is C^∞ would be unnecessary.

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